From the last term of (1) in a similar manner

$$\lim_{n=\infty} n \sin^{-1}\left[\frac{a\tan\dfrac{\pi}{n}}{(R^2-a^2)^{1/2}}\right] = \frac{\pi a}{(R^2-a^2)^{1/2}}$$

Substituting these values in (1) and reducing

$$V_{\text{II}} = \lim_{n=\infty} V = \tfrac{4}{3}\pi[R^3 - (R^2-a^2)^{3/2}]$$

which is the result for the familiar problem of the volume cut from a sphere of radius $R$ by circular cylinder of radius $a$ when the center of the sphere is on the axis of the cylinder.

Also solved by PAUL CAPRON, A. W. SMITH and the PROPOSER.

**401. Proposed by LAENAS G. WELD, Pullman, Ill.**

Given a continuum of triangles whose sides are in arithmetical progression, the common difference being $h$: (a) The ratio of the mean value of all the triangles, the mean of whose three sides is not greater than $\mu$, to the area of the triangle, the mean of whose three sides is equal to $\mu$, is $(\mu + 2h)/3\mu$. Indicate the limiting values of this ratio and show that, when it is equal to 1/2, the triangle whose mean side is $\mu$ is right angled. (b) The ratio of the mean value of the areas of the circles inscribed in all these triangles, the mean of whose three sides is not greater than $\mu$, to that of the circle inscribed in the triangle, the mean of whose three sides is equal to $\mu$, has the limiting values 1/2 and 1/3. When the triangle whose mean side is $\mu$ is right angled, the ratio in question is 4/9. (c) Of the circles circumscribed about these triangles the minimum has the radius $2h$.

## SOLUTION BY A. H. WILSON, Haverford College.

Let $x - h$, $x$, and $x + h$ be the three sides. Then the area is

$$[s(s-a)(s-b)(s-c)]^{1/2} = \tfrac{1}{4}[3x^2(x^2-4h^2)]^{1/2}.$$

The mean area of all triangles, the mean of whose three sides is not greater than $\mu$, is

$$\frac{\sqrt{3}}{4(\mu-2h)}\int_{2h}^{\mu} x\sqrt{x^2-4h^2}\,dx = \frac{\sqrt{3}}{12}\sqrt{\mu^2-4h^2}(\mu+2h),$$

the least value of $x$ being $2h$.

The area of the triangle for which $x = \mu$ is

$$\tfrac{1}{4}[3\mu^2(\mu^2-4h^2)]^{1/2} = \frac{\sqrt{3}}{4}\mu(\mu^2-4h^2)^{1/2}.$$

(a) The ratio of these two areas is $(\mu + 2h)/3\mu$. The limits of this ratio occur for the values $\mu = 2h$ and $\mu = \infty$, and are, respectively, 2/3 and 1/3. If the ratio is equal to 1/2, then $\mu = 4h$; and the sides $3h$, $4h$, and $5h$ are those of a right triangle.

(b) The area of the inscribed circle is

$$\pi r^2 = \pi(x^2-4h^2)/12. \quad (r = [(s-a)(s-b)(s-c)/s]^{1/2}).$$

The mean of such areas for triangles of the first class is

$$\frac{\pi}{12(\mu-2h)}\int_{2h}^{\mu}(x^2-4h^2)\,dx = \frac{\pi}{36}(\mu^2+2h\mu-8h^2).$$

The area of the circle of the second class is $\pi(\mu^2 - 4h^2)/12$; and the ratio of the two areas is $(\mu + 4h)/3(\mu + 2h)$. The limiting values of this ratio are 1/2 and 1/3, and for the value 4/9 the mean side is $4h$, and the triangle is right-angled.

(c) The area of the circumscribed triangle is $\pi R^2 = \pi(abc)^2/16s(s-a)(s-b)(s-c) = \pi(x^2-h^2)^2/3(x^2-4h^2)$. For a minimum, equate the derivative to 0; and there results for $x$ the value $\sqrt{7}h$, giving a minimum. For this value of $x$ the radius of the circumscribed circle is $2h$.

Also solved by ELIJAH SWIFT.